Dressed-particle approach in the nonrelativistic classical limit

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(Received 28 October 2008; published 26 February 2009)

For a nonrelativistic classical particle undergoing arbitrary oscillations in external fields, the generalized effective potential Ψ is derived through calculating the nonlinear eigenfrequencies of the particle-field system. Specifically, the ponderomotive potential is extended to a nonlinear oscillator, resulting in multiple branches near the primary resonance. For a pair of particle natural frequencies in a beat resonance, Ψ scales linearly with the internal actions and is analogous to the dipole potential for a two-level quantum system. Thus cold quantum particles and highly excited quasiclassical objects permit uniform manipulation tools, particularly, one-way walls.

DOI: 10.1103/PhysRevE.79.026407 PACS number(s): 52.35.Mw, 05.45.−a, 45.20.Jj, 45.05.+x

I. INTRODUCTION

Multiscale adiabatic dynamics of classical particles in oscillating and static fields is simplified within the oscillation-center (OC) approach, which allows separating fast quiver motion of the particles from their slow translational motion [1–3]. Hence the average forces are embedded into the properties of the OC, yielding a quasiparticle with a variable effective mass $m_{\rm eff}$ [4,5]. In each given case, $m_{\rm eff}$ can be Taylor-expanded at nonrelativistic energies so as to appear as an effective potential Ψ [4], e.g., ponderomotive [6–9] or diamagnetic [10]. Yet the nonrelativistic limit must permit also an independent calculation of Ψ. For linear oscillations, the generalized effective potential was derived in Ref. [3]. However, a comprehensive method of finding Ψ for nonlinear quiver motion has not been proposed.

The purpose of this work is to calculate, from first principles, the generalized effective potential Ψ for a nonrelativistic classical particle undergoing arbitrary oscillations in high-frequency or static fields. We proceed by finding eigenmodes in the particle-field system; hence Ψ is obtained like in the dressed-atom approach [11–14] but from nonlinear classical equations. Specifically, we show that the ponderomotive potential extended to a nonlinear oscillator has multiple branches near the primary resonance. We also show that, for a pair of natural frequencies in a beat resonance, Ψ scales linearly with the internal actions and is analogous to the dipole potential for a two-level quantum system. Thus cold quantum particles and highly excited quasiclassical objects permit uniform manipulation tools, particularly, stationary asymmetric barriers, or one-way walls [15–24].

The work is organized as follows. In Sec. II, we obtain the general form of the effective potential Ψ. In Sec. III, we derive the equations for oscillation modes. In Sec. IV, we calculate the ponderomotive potential from the infinitesimal frequency shift of the oscillating field coupled to a particle at a primary resonance, both linear and nonlinear; see also the Appendix. In Sec. V, we find Ψ near a beat resonance and show the analogy with the dipole potential. In Sec. VI, we explain how Ψ allows one-way walls. In Sec. VII, we summarize our main results.

II. GENERALIZED EFFECTIVE POTENTIAL

Consider a classical particle undergoing adiabatic oscillations in arbitrary external fields. Mapping out the quiver dynamics yields that the slow, OC motion is governed by the Lagrangian (see Appendix A in Ref. [4]),

$$\mathcal{L}_0 = \langle L\rangle - \boldsymbol{\varpi}\cdot\boldsymbol{\mathcal{J}}, \tag{1}$$

where $\langle L\rangle$ is the true particle time-averaged Lagrangian, $\boldsymbol{\varpi}\equiv\dot{\boldsymbol{\vartheta}}$ is the canonical frequency vector, and $(\boldsymbol{\vartheta},\boldsymbol{\mathcal{J}})$ are the angle-action variables of the particle *free* oscillations, if any, with $\boldsymbol{\mathcal{J}}$=const. Suppose that $\langle L\rangle$ depends on the OC coordinate $\mathbf{r}$ only through the oscillation parameters $\mathfrak{P}$ [25]; hence $\mathcal{L}_0$ can be calculated, to the leading order, as if $\mathfrak{P}$ were fixed. Yet in the latter case there would be no average force, so in the OC comoving frame (henceforth denoted by prime) $\mathcal{L}_0'$ must be a constant determined by $\mathfrak{P}'$ and $\boldsymbol{\mathcal{J}}$. Assuming the gauge such that $L'=-mc^2$ at zero quiver velocity, this constant is defined uniquely, like for a true particle [26], and can be represented as $\mathcal{L}_0'=-m_{\rm eff}c^2$, c being the speed of light. Accounting for the time dilation, the OC Lagrangian in the laboratory frame then reads

$$\mathcal{L}_0 = -\, m_{\rm eff}c^2\sqrt{1-v^2/c^2}, \tag{2}$$

where $\mathbf{v}\equiv\dot{\mathbf{r}}$ is the OC velocity, $m_{\rm eff}(\mathfrak{P}';\boldsymbol{\mathcal{J}})$ can be understood as the effective mass, $\mathfrak{P}'=\mathfrak{P}'(\mathbf{r},\mathbf{v})$, and the dependence on $\boldsymbol{\mathcal{J}}$ is parametric.

Assume nonrelativistic dynamics, i.e., $v\ll c$ and $\delta m\equiv m_{\rm eff}-m\ll m$, where m is the true mass; then

$$\mathcal{L}_0 = \tfrac{1}{2}mv^2 - \mathcal{U}, \quad \mathcal{U}=\delta mc^2. \tag{3}$$

Hence the effective potential can be found by calculating δm from the relativistic particle trajectory in given fields, as shown in Ref. [4]. However, a general nonrelativistic approach is also possible as follows. Consider the OC extended space including the angles $\boldsymbol{\vartheta}$. For that, the Lagrangian equivalent to Eq. (3) can be written as

$$\mathcal{L} = \mathcal{L}_0(\mathbf{r},\mathbf{v}) + \dot{\boldsymbol{\vartheta}}\cdot\boldsymbol{\mathcal{J}}, \tag{4}$$

so the added Euler-Lagrange equation correctly describes the adiabatic oscillations, reading $\dot{\boldsymbol{\mathcal{J}}}=0$. Hence the corresponding Hamiltonian $\mathcal{H}(\mathbf{r},\mathbf{P};\boldsymbol{\vartheta},\boldsymbol{\mathcal{J}})$ equals

$$\mathcal{H}=\frac{P^2}{2m}+\Psi, \quad \Psi=\mathcal{U}-\frac{1}{2m}(\partial_{\mathbf{v}}\mathcal{U})^2, \tag{5}$$

where $\mathbf{P}=m\mathbf{v}-\partial_{\mathbf{v}}\mathcal{U}$ is the canonical momentum.

We now require that the variables $(\boldsymbol{\vartheta},\boldsymbol{\mathcal{J}})$ include oscillating field modes to which the particle is coupled. Then, like in the dressed-atom approach [11,12], Ψ depends on $\mathbf{r}$ and $\mathbf{P}$ only parametrically, through the eigenfrequencies $\boldsymbol{\varpi}(\mathbf{r},\mathbf{P},\boldsymbol{\mathcal{J}})$. Correspondingly, $\Psi(\boldsymbol{\mathcal{J}}=0)=0$, and one gets from $\boldsymbol{\varpi}=\partial_{\boldsymbol{\mathcal{J}}}\mathcal{H}$ that

$$\Psi=\int \boldsymbol{\varpi}\cdot d\boldsymbol{\mathcal{J}}. \tag{6}$$

Thus finding the effective potential Ψ is equivalent to deriving the eigenspectrum $\boldsymbol{\varpi}$ of the particle-field system.

The canonical frequencies can be redefined such that $\boldsymbol{\varpi}\to\boldsymbol{\varpi}+\text{const}$, adding a constant to Ψ. Albeit arbitrarily large, this contribution does not affect the motion equations, so we abandon the requirement that Ψ must remain small compared to mc^2; hence actual physical frequencies can be used for $\boldsymbol{\varpi}$. Specifically, for uncoupled linear modes one gets from Eq. (6) that $\Psi=\Psi_0$,

$$\Psi_0=\boldsymbol{\Omega}\cdot\mathbf{J}+\boldsymbol{\omega}\cdot\mathbf{I}, \tag{7}$$

where $(\boldsymbol{\Omega},\mathbf{J})$ and $(\boldsymbol{\omega},\mathbf{I})$ stand for the frequencies and the actions of the particle and the field partial oscillations, correspondingly. For an unbounded field $(I\to\infty)$, the second term in Eq. (7) is infinite; as it is fixed though, the force on the OC is determined only by $\boldsymbol{\Omega}\cdot\mathbf{J}$. On the other hand, particle-field coupling would produce yet another term, $\Phi\equiv\Psi-\Psi_0$. To show how it also flows from Eq. (6) is the purpose of the next sections.

III. PARTIAL MODE DECOMPOSITION

Suppose there are weakly nonlinear oscillations $\boldsymbol{\xi}(t)$, both of the particle [3] and of external fields [27], so their Lagrangian reads $\tilde{\mathcal{L}}(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})=\tilde{\mathcal{L}}_0+\tilde{\mathcal{L}}_{\text{int}}$, where $\tilde{\mathcal{L}}_{\text{int}}$ is a perturbation to a bilinear form $\tilde{\mathcal{L}}_0$ [28],

$$\tilde{\mathcal{L}}_0=\tfrac{1}{2}(\dot{\boldsymbol{\xi}}\cdot\hat{M}\dot{\boldsymbol{\xi}})-(\dot{\boldsymbol{\xi}}\cdot\hat{R}\boldsymbol{\xi})-\tfrac{1}{2}(\boldsymbol{\xi}\cdot\hat{Q}\boldsymbol{\xi}). \tag{8}$$

Here $\hat{M}$, $\hat{R}$, $\hat{Q}$ are $N\times N$ real matrices; $\hat{M}$ and $\hat{Q}$ are symmetric, $\hat{R}$ is antisymmetric, and $\operatorname{rank}\hat{M}=N\equiv\dim\boldsymbol{\xi}$. At zero $\hat{R}$, $\tilde{\mathcal{L}}_0(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})$ can be diagonalized to yield

$$\tilde{\mathcal{L}}_0=\sum_{j=1}^{N}L_j, \quad L_j=\frac{1}{2}M_j\dot{\xi}_j^2-\frac{1}{2}Q_j\xi_j^2, \tag{9}$$

where L_j describe individual modes ξ_j [29]. Then

$$\hat{D}_j\xi_j=\delta_{\xi_j}\tilde{\mathcal{L}}_{\text{int}}, \quad \hat{D}_j=M_jd_t^2+Q_j, \tag{10}$$

δ and d_t standing for the variational and time derivatives. Yet, such decomposition does not hold in the general case, so we redefine eigenmodes, following Ref. [30].

Extend the configuration space by introducing

$$\langle\ell|=(-\boldsymbol{\pi}\hat{M}^{-1},\boldsymbol{\xi}), \quad |r\rangle=(\boldsymbol{\xi},\hat{M}^{-1}\boldsymbol{\pi})^{\mathrm{T}} \tag{11}$$

(the index T denotes matrix transpose) as the new, "left" and "right," coordinate vectors, where $\boldsymbol{\pi}=\hat{M}\dot{\boldsymbol{\xi}}-\hat{R}\boldsymbol{\xi}$ is the old canonical momentum. Then

$$\tilde{\mathcal{L}}_0=\tfrac{1}{4}[\langle\dot{\ell}|\hat{\mathbb{M}}|r\rangle-\langle\ell|\hat{\mathbb{M}}|\dot{r}\rangle]+\tfrac{1}{2}\langle\ell|\hat{\mathbb{F}}|r\rangle, \tag{12}$$

where we omitted a full time derivative and introduced

$$\hat{\mathbb{M}}=\begin{pmatrix}\hat{M} & 0\\ 0 & \hat{M}\end{pmatrix}, \quad \hat{\mathbb{F}}=\begin{pmatrix}\hat{R} & \hat{M}\\ \hat{F} & \hat{R}\end{pmatrix}, \tag{13}$$

with $\hat{F}=\hat{R}\hat{M}^{-1}\hat{R}-\hat{Q}$. Thus the resulting equations are

$$\langle\dot{\ell}|\hat{\mathbb{M}}+\langle\ell|\hat{\mathbb{F}}=0, \quad \hat{\mathbb{M}}|\dot{r}\rangle-\hat{\mathbb{F}}|r\rangle=0, \tag{14}$$

both equivalent to

$$\hat{M}\ddot{\boldsymbol{\xi}}-2\hat{R}\dot{\boldsymbol{\xi}}+\hat{Q}\boldsymbol{\xi}=0. \tag{15}$$

Equation (15) has $2N$ eigenmodes $\boldsymbol{\xi}_j=\bar{\boldsymbol{\xi}}_je^{-i\nu_jt}$, with ν_j hence assumed real and nonzero; therefore, for each $\boldsymbol{\xi}_j$, there also exists a mode $\boldsymbol{\xi}_{-j}=\boldsymbol{\xi}_j^*$, and $\bar{\boldsymbol{\xi}}_j$ are generally not orthogonal. The corresponding eigenmodes of Eqs. (14) are

$$\langle\ell_j|=e^{i\nu_jt}\langle\bar{\ell}_j|, \quad |r_j\rangle=e^{-i\nu_jt}|\bar{r}_j\rangle, \tag{16}$$

with vector amplitudes

$$\langle\bar{\ell}_j|=(-i\nu_j\bar{\boldsymbol{\xi}}_j^*-\bar{\boldsymbol{\xi}}_j^*\hat{R}\hat{M}^{-1},\bar{\boldsymbol{\xi}}_j^*), \tag{17}$$

$$|\bar{r}_j\rangle=(\bar{\boldsymbol{\xi}}_j,-i\nu_j\bar{\boldsymbol{\xi}}_j-\hat{M}^{-1}\hat{R}\bar{\boldsymbol{\xi}}_j)^{\mathrm{T}}, \tag{18}$$

and $\hat{\mathbb{M}}_{jk}\equiv\langle\bar{\ell}_j|\hat{\mathbb{M}}|\bar{r}_k\rangle=-2i\rho_{jk}$, where

$$\rho_{jk}=\tfrac{1}{2}\bar{\boldsymbol{\xi}}_j^*\cdot[(\nu_j+\nu_k)\hat{M}-2i\hat{R}]\cdot\bar{\boldsymbol{\xi}}_k. \tag{19}$$

The matrix $\hat{\rho}$ is diagonal for distinct ν_j, as seen from Eq. (15), or can be diagonalized when some of the frequencies coincide [30]; thus

$$\hat{\mathbb{M}}_{jk}=-2i\rho_j\delta_{jk}, \quad \rho_j\equiv\nu_j(\bar{\boldsymbol{\xi}}_j^*\cdot\hat{M}\bar{\boldsymbol{\xi}}_j), \tag{20}$$

$\rho_j=-\rho_{-j}$. (Hence modes with $\nu_j=0$ are orthogonal to the others and can be considered separately, as implied below.) Therefore any $\langle\ell|$ and $|r\rangle$ are decomposed as

$$\langle\ell|=\sum_{j=-N}^{N}{}'\,\ell_j\langle\bar{\ell}_j|, \quad |r\rangle=\sum_{j=-N}^{N}{}'\,r_j|\bar{r}_j\rangle, \tag{21}$$

where the primes stand for skipping $j=0$, and

$$\ell_j=\frac{i}{2\rho_j}\langle\ell|\hat{\mathbb{M}}|\bar{r}_j\rangle, \quad r_j=\frac{i}{2\rho_j}\langle\bar{\ell}_j|\hat{\mathbb{M}}|r\rangle. \tag{22}$$

Since $\langle\ell|$ and $|r\rangle$ are real, one has $r_j=\ell_j^*\equiv\psi_j\sqrt{2}$ and $\psi_{-j}=\psi_j^*$; hence Eq. (9) is recovered, but with

$$L_j=\frac{i\rho_j}{2}(\dot{\psi}_j\psi_j^*-\psi_j\dot{\psi}_j^*)-\rho_j\nu_j|\psi_j|^2. \tag{23}$$

The resulting equations for individual modes are

$$\hat{D}_j\psi_j = \delta_{\psi_j^*}\tilde{\mathcal{L}}_{\text{int}}, \quad \hat{D}_j = \rho_j(\nu_j - i d_t), \tag{24}$$

similar to reduced Eqs. (10). Particularly, at zero $\tilde{\mathcal{L}}_{\text{int}}$,

$$\psi_j = \sqrt{2\mathcal{J}_j}e^{-i\vartheta_j}, \quad \dot{\vartheta}_j = \nu_j, \quad \mathcal{J}_j = \text{const}. \tag{25}$$

On the other hand, $L_j=(\dot{\vartheta}_j-\nu_j)\mathcal{J}_j$; thus $\partial_{\dot{\vartheta}_j}L_j=\mathcal{J}_j$ is also the action corresponding to the angle ϑ_j:

$$\mathcal{J}_j = \rho_j|\psi_j|^2, \tag{26}$$

so ν_j is the canonical frequency. Then the mode energy is $\nu_j\mathcal{J}_j$ (thus $\rho_j>0$ for stable modes with $\nu_j>0$, henceforth implied), and Eq. (7) is recovered.

Below we apply Eqs. (24) to find eigenmodes for nonzero $\tilde{\mathcal{L}}_{\text{int}}$, with $\boldsymbol{\psi}$ becoming partial oscillations; hence the effective potential is obtained from Eq. (6).

IV. PRIMARY RESONANCE

A. Linear oscillator

First, we calculate Ψ for a linear coupling between a pair of modes ψ_1 and ψ_2, say,

$$\tilde{\mathcal{L}}_{\text{int}} = \sigma\psi_1\psi_2^* + \sigma^*\psi_1^*\psi_2, \tag{27}$$

where σ=const. In this case, Eqs. (24) read

$$\hat{D}_1\psi_1 = \sigma^*\psi_2, \quad \hat{D}_2\psi_2 = \sigma\psi_1. \tag{28}$$

Then the eigenfrequencies ϖ are governed by

$$\rho_1\rho_2(\varpi - \nu_1)(\varpi - \nu_2) = |\sigma|^2, \tag{29}$$

yielding two roots $\varpi_{1,2}$ independent of $\mathcal{J}_{1,2}$:

$$\varpi_{1,2} = \frac{\nu_1+\nu_2}{2} \pm \sqrt{\left(\frac{\nu_1-\nu_2}{2}\right)^2 + \frac{|\sigma|^2}{\rho_1\rho_2}}. \tag{30}$$

Hence one obtains

$$\Psi = \varpi_1\mathcal{J}_1 + \varpi_2\mathcal{J}_2. \tag{31}$$

As a particular case, consider interaction of a particle internal mode having unperturbed frequency Ω and action $J=\rho|\psi|^2$ with an external oscillating field $E=\bar{E}e^{-i\omega t}$ having unperturbed frequency ω and action $I=\rho_E|E|^2$. Given that the field occupies a volume $V\to\infty$, the frequency shifts $\delta\Omega$ and $\delta\omega$ due to coupling are infinitesimal, so Eq. (30) yields

$$\delta\Omega\rho = \frac{|\sigma|^2}{(\Omega-\omega)\rho_E}, \quad \delta\omega\rho_E = \frac{|\sigma|^2}{(\omega-\Omega)\rho}. \tag{32}$$

Since $\rho_E\propto V$, one has $\delta\Omega J \ll \delta\omega I$, whereas

$$\delta\omega\, I = \frac{|\sigma|^2}{(\omega-\Omega)\rho}|\bar{E}|^2 \tag{33}$$

is nonvanishing. Then, from Eq. (31), one gets

$$\Psi = \Omega J + \Phi_0, \quad \Phi_0 = -\tfrac{1}{4}\alpha|\bar{E}|^2, \tag{34}$$

where Φ_0 is the so-called ponderomotive potential (for the general expression, see the Appendix), an insignificant constant ωI is removed, and $\alpha=4|\sigma|^2[(\Omega-\omega)\rho]^{-1}$. Since

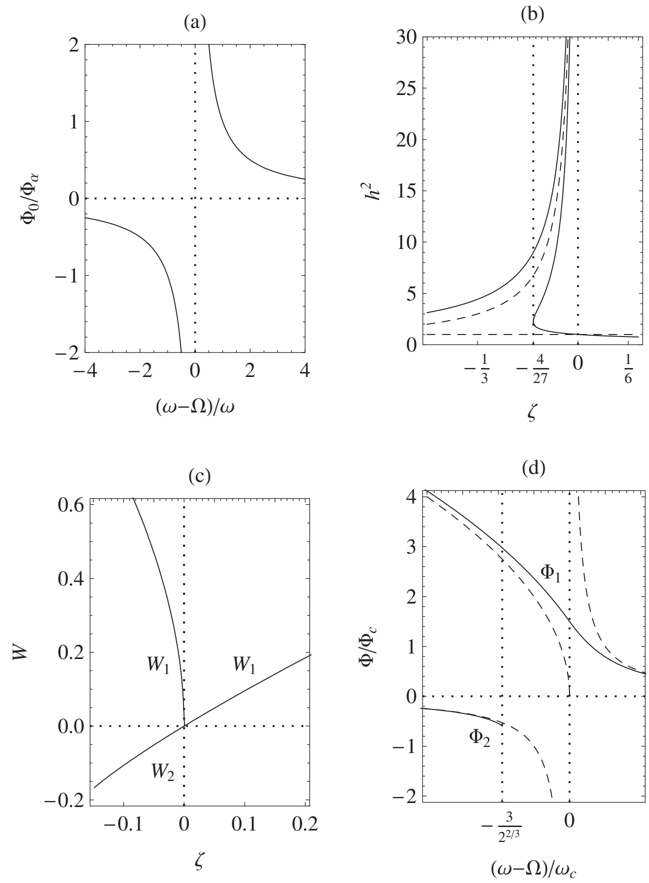

FIG. 1. (a) Resonant ponderomotive potential Φ_0 [Eq. (35)] in units $\Phi_\alpha\equiv\kappa_\alpha^2|\bar{E}|^2/\omega$ vs $\omega-\Omega$ in units ω. (b) Solid: squared normalized frequency shift $h=\delta\omega I/\Phi_0$ vs $\zeta=\beta\Phi_0/(\gamma\rho)^2$, $\zeta(J=0)\propto(\omega-\Omega)^{-3}$; dashed: approximations $h^2=1$ and $h^2=|\zeta|^{-1}$. (c) $W(\zeta)$ [Eq. (42)]; consists of branches $W_1>0$ and $W_2<0$. (d) Effective potential Φ in units $\Phi_c\equiv\rho^2\omega_c^2/\beta=\frac{2}{3}\Phi(0)$ vs $\omega-\Omega$ in units $\omega_c\equiv\beta^{1/3}|\sigma\bar{E}|^{2/3}/\rho$, assuming $J=0$ and $\beta>0$. Solid: Eq. (42) (consists of branches $\Phi_1>0$ and $\Phi_2<0$); dashed: Eqs. (43) and (44).

$$\Phi_0 = \frac{\kappa_\alpha^2|\bar{E}|^2}{\omega-\Omega}, \tag{35}$$

where $\kappa_\alpha^2\equiv|\sigma|^2/\rho>0$, the effective potential becomes infinite at the linear resonance [Fig. 1(a)]. However, nonlinear effects remove this singularity, as we show below.

B. Nonlinear oscillator

Consider the effective potential near a nonlinear resonance, with a Duffing oscillator as a model system. Then

$$\tilde{\mathcal{L}}_{\text{int}} = \sigma\psi E^* + \sigma^*\psi^* E + \tfrac{1}{2}\beta|\psi|^4, \tag{36}$$

where β=const, yielding

$$\hat{D}\psi = \sigma^* E + \beta|\psi|^2\psi, \quad \hat{D}_E E = \sigma\psi. \tag{37}$$

Separate the driven motion from free oscillations, $\psi=Xe^{-i(\omega+\delta\omega)t}+Ye^{-i(\Omega+\delta\Omega)t}$, so

$$-\delta\omega\,\rho_E\bar{E}=\sigma X, \tag{38}$$

$$(\Omega-\omega)\rho X=\sigma^*\bar{E}+2\beta|Y|^2X+\beta|X|^2X, \tag{39}$$

$$-\delta\Omega\,\rho Y=\beta|Y|^2Y+2\beta|X|^2Y. \tag{40}$$

From Eq. (40), it follows that $\delta\Omega J\sim\beta|XY|^2$, which we assume, for simplicity, small compared to $\Phi_0\sim\delta\omega I$; hence $\Psi\approx\Omega J+\Phi$, where $\Phi=\int_0^I\delta\omega\,dI$ is the modified ponderomotive potential. The field frequency shift $\delta\omega$ is found from Eqs. (38) and (39), which yield a cubic equation for $h=\delta\omega I/\Phi_0$:

$$(1+\zeta h^2)h=1, \tag{41}$$

where $\zeta=\beta\Phi_0/(\gamma\rho)^2$, $\Phi_0=|\sigma\bar{E}|^2/(\gamma\rho)$, $\gamma=\omega-\Omega+\varsigma$ is the detuning frequency, $\varsigma=2\beta J/\rho^2$ is the nonlinear shift of the resonance frequency, and the above condition of negligible $\delta\Omega J$ reads $\varsigma/(\omega-\Omega)\ll1$. Thus

$$\Phi=\frac{\gamma^2\rho^2}{\beta}W(\zeta),\quad W(\zeta)=\int_0^{\zeta}h(\tilde{\zeta})d\tilde{\zeta}, \tag{42}$$

where $h(\zeta)$ has either one or three branches [Fig. 1(b)]. In the latter case, realized at $-\frac{4}{27}<\zeta<0$, one of those is unstable [31]; hence two actual branches of Φ.

The asymptotic form of the ponderomotive potential is found as follows. At $|\zeta|\ll1$, Eq. (41) yields $h\approx1$, for either sign of ζ, and additional roots specific for $\zeta<0$ read $h\approx\pm|\zeta|^{-1/2}$. Among the latter two, $h>0$ is the one that is unstable, so we keep only the remaining solutions. After renumbering them accordingly, one gets from Eq. (42) that $W_1^+\approx\zeta$, $W_2^-\approx\zeta$, and $W_1^-\approx2\sqrt{|\zeta|}$, where $\pm$ denotes $\zeta\to0\pm$ [Fig. 1(c)]. Hence approximate expressions follow for a large detuning $\gamma\propto|\zeta|^{-1/3}\,\mathrm{sgn}\zeta$:

$$\Phi_1^+\approx\Phi_2^-\approx\Phi_0=\frac{\kappa_\alpha^2|\bar{E}|^2}{\gamma}, \tag{43}$$

$$\Phi_1^-\approx2\kappa_\alpha\rho|\bar{E}|\frac{\sqrt{|\gamma\beta|}}{\beta}, \tag{44}$$

and Eq. (35) is recovered from Eq. (43) at small ζ and $\varsigma=0$. On the other hand, at large ζ, W_2 does not exist, whereas $h(\zeta\to\pm\infty)\approx|\zeta|^{-1/3}\,\mathrm{sgn}\zeta$ yields $W_1\approx\frac{3}{2}|\zeta|^{2/3}$ at $\zeta\to\pm\infty$, or $\gamma\to0\pm$; thus W_1 is continuous near the resonance. Hence one gets from Eq. (42) that $\Phi_1(\gamma\to0)\approx3|\sigma\bar{E}|^{4/3}|\beta|^{2/3}/(2\beta)$, i.e., the singularity vanishes [Fig. 1(d)].

V. BEAT RESONANCE

Suppose that $\tilde{\mathcal{L}}_{\mathrm{int}}$ also contains a term cubic in $\boldsymbol{\psi}$, say,

$$\delta\tilde{\mathcal{L}}_{\mathrm{int}}=\epsilon\psi_1\psi_2^*\psi_3^*+\epsilon^*\psi_1^*\psi_2\psi_3, \tag{45}$$

where ϵ=const. That would normally yield a potential small compared to Φ_0, yet maybe except when

$$\nu_1\approx\nu_2+\nu_3. \tag{46}$$

Below we study the beat resonance (46) neglecting the quadratic $\tilde{\mathcal{L}}_{\mathrm{int}}$, so the envelopes $\bar{\psi}_j=\psi_je^{i\nu_jt}$ satisfy

$$-i\rho_1\dot{\bar{\psi}}_1=\epsilon^*\bar{\psi}_2\bar{\psi}_3e^{-i\Delta t}, \tag{47}$$

$$-i\rho_2\dot{\bar{\psi}}_2=\epsilon\bar{\psi}_1\bar{\psi}_3^*e^{i\Delta t}, \tag{48}$$

$$-i\rho_3\dot{\bar{\psi}}_3=\epsilon\bar{\psi}_1\bar{\psi}_2^*e^{i\Delta t}, \tag{49}$$

where $\Delta\equiv\nu_3+\nu_2-\nu_1$ is the detuning frequency.

A. Linear coupling

Suppose that Δ is large enough, so Eqs. (47)–(49) can be solved by averaging. Then split $\bar{\psi}_j$ into driven and free motion: $\bar{\psi}_j=X_je^{-i\eta_jt}+Y_je^{-i\delta\nu_jt}$, where $\delta\nu_j\ll\eta_j$ are the nonlinear frequency shifts, and

$$\eta_1=\delta\nu_2+\delta\nu_3+\Delta, \tag{50}$$

$$\eta_2=\delta\nu_1-\delta\nu_3-\Delta, \tag{51}$$

$$\eta_3=\delta\nu_1-\delta\nu_2-\Delta. \tag{52}$$

Thus the corresponding equations read

$$-\eta_1\rho_1X_1=\epsilon^*Y_2Y_3,\quad -\delta\nu_1\rho_1Y_1=\epsilon^*(Y_2X_3+X_2Y_3),$$

$$-\eta_2\rho_2X_2=\epsilon Y_1Y_3^*,\quad -\delta\nu_2\rho_2Y_2=\epsilon(Y_1X_3^*+X_1Y_3^*),$$

$$-\eta_3\rho_3X_3=\epsilon Y_1Y_2^*,\quad -\delta\nu_3\rho_3Y_3=\epsilon(Y_1X_2^*+X_1Y_2^*),$$

where insignificant oscillating terms are neglected. Since $\delta\nu_j\ll\Delta$, one has

$$\delta\nu_1=-\frac{\mathcal{J}_2+\mathcal{J}_3}{\Gamma\Delta},\quad \delta\nu_2=\frac{\mathcal{J}_3-\mathcal{J}_1}{\Gamma\Delta},\quad \delta\nu_3=\frac{\mathcal{J}_2-\mathcal{J}_1}{\Gamma\Delta},$$

where $\Gamma\equiv\rho_1\rho_2\rho_3/|\epsilon|^2>0$. Then, from Eqs. (6), one gets $\Psi=\Psi_0+\Phi$,

$$\Phi=\frac{\mathcal{J}_1\mathcal{J}_2+\mathcal{J}_1\mathcal{J}_3-\mathcal{J}_2\mathcal{J}_3}{(\nu_1-\nu_2-\nu_3)\Gamma}. \tag{53}$$

Particularly, if the third mode corresponds to a macroscopic oscillating field tuned close to the beat resonance

$$\omega\approx\Omega_1-\Omega_2, \tag{54}$$

then $J_{1,2}\ll I\propto|\bar{E}|^2$. Hence the "hybrid" ponderomotive potential that is obtained is simultaneously proportional to $|\bar{E}|^2$ and the internal actions J_j:

$$\Phi=\frac{\kappa_\epsilon^2}{\Delta}(J_2-J_1)|\bar{E}|^2, \tag{55}$$

where $\Delta=\omega-(\Omega_1-\Omega_2)$, and $\kappa_\epsilon^2\equiv|\epsilon|^2/(\rho_1\rho_2)>0$.

B. Nonlinear coupling

At $\Delta\lesssim\delta\nu_j$, the internal mode equations

$$-i\rho_1\dot{\bar{\psi}}_1=\epsilon^*\bar{\psi}_2\bar{E}e^{-i\Delta t},\quad -i\rho_2\dot{\bar{\psi}}_2=\epsilon\bar{\psi}_1\bar{E}^*e^{i\Delta t} \tag{56}$$

do not allow averaging, so the above derivation is modified as follows. First, consider strictly periodic oscillations, in which case $\bar{\boldsymbol{\psi}} \equiv (\bar{\psi}_1, \bar{\psi}_2)^{\mathrm{T}}$ rewrites as $\bar{\boldsymbol{\psi}} = \hat{T}_\Delta \hat{C}_\psi^{-1} \mathbf{x}$, where $\hat{T}_y = \mathrm{diag}(e^{-iyt/2}, e^{iyt/2})$, and $\hat{C}_\psi = \hat{T}_{-\phi}\, \mathrm{diag}(\rho_1, \rho_2)^{1/2}$, $\phi = \arg(\epsilon \bar{E}^*) + \pi$. Then

$$i\dot{x}_1 = \frac{\varepsilon}{2}x_2 - \frac{\Delta}{2}x_1, \quad i\dot{x}_2 = \frac{\varepsilon}{2}x_1 + \frac{\Delta}{2}x_2, \tag{57}$$

where $\varepsilon = 2\kappa_\epsilon |\bar{E}|$. Equations (57) yield two eigenmodes at frequencies $\pm\frac{1}{2}\Lambda$, $\Lambda = (\varepsilon^2 + \Delta^2)^{1/2}$, so $\mathbf{x} = \hat{U}\hat{T}_\Lambda \bar{\mathbf{x}}$, where

$$\hat{U} = \begin{pmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{pmatrix}, \tag{58}$$

$\bar{\mathbf{x}} \equiv (\bar{x}_+, \bar{x}_-)^{\mathrm{T}}$ is constant, and Θ satisfies

$$\cos 2\Theta = -\Delta/\Lambda, \quad \sin 2\Theta = \varepsilon/\Lambda. \tag{59}$$

Then

$$\boldsymbol{\psi} = \hat{T}_\omega \hat{S} \boldsymbol{\chi}, \tag{60}$$

where $\hat{S} \equiv \hat{C}_\psi^{-1}\hat{U}\hat{C}_\chi$ is a constant matrix, and $\hat{C}_\chi$ is an arbitrary diagonal matrix introduced for uniformity. Hence we take $\hat{C}_\chi = \mathrm{diag}(\rho_+, \rho_-)^{1/2}$, where $\rho_\pm \equiv \varpi_\pm$,

$$\varpi_\pm = \tfrac{1}{2}(\Omega_1 + \Omega_2) \pm \tfrac{1}{2}\Lambda, \tag{61}$$

$\chi_\pm = \bar{\chi}_\pm e^{-i\vartheta_\pm}$, $\dot{\vartheta}_\pm = \varpi_\pm$, and $\bar{\boldsymbol{\chi}} = \hat{C}_\chi^{-1}\bar{\mathbf{x}}$.

When the oscillation parameters evolve, treat Eq. (60) as a formal change of variables for Eq. (23). Then $\tilde{\mathcal{L}} = L_+ + L_-$, the field part being omitted, and

$$L_\pm = \frac{i\rho_\pm}{2}(\dot{\chi}_\pm \chi_\pm^* - \dot{\chi}_\pm^* \chi_\pm) - \rho_\pm \nu_\pm |\chi_\pm|^2. \tag{62}$$

Hence $\chi_\pm$ are independent linear modes with frequencies $\varpi_\pm$ and conserved actions $J_\pm = \rho_\pm |\chi_\pm|^2$ (Sec. III), and Eq. (6) yields

$$\Psi = \frac{1}{2}(\Omega_1 + \Omega_2)(J_+ + J_-) + \frac{\Lambda}{2}(J_+ - J_-). \tag{63}$$

At $\Delta \gg \varepsilon$, $J_\pm = J_{1,2}$ for $\Delta < 0$ and $J_\pm = J_{2,1}$ for $\Delta > 0$, so Eq. (55) is recovered by Taylor expansion of Eq. (63).

C. Quantum analogy

The above classical particle is the limit of a quantum system with plentiful states coupled to the field simultaneously and Ω_j being the unperturbed transition frequencies [Fig. 2(a)]. Yet, with $\rho_j \to \hbar$, Eqs. (56) are also equivalent to those describing a *two*-level system [13,14,32], with the unperturbed eigenfrequencies Ω_1 and Ω_2 and the Rabi frequency $\Omega_{\mathrm{R}} = \varepsilon$ [Fig. 2(b)]. Hence Eq. (63) yields as well the dipole potential for a two-level quantum object, e.g., a cold atom ([11], pp. 454–461), [12]:

$$\Psi = \frac{\hbar}{2}(\Omega_1 + \Omega_2) + \frac{\hbar}{2}(n_+ - n_-)\sqrt{\Delta^2 + \Omega_{\mathrm{R}}^2}, \tag{64}$$

where $n_j = |\chi_j|^2$ are the occupation numbers ($J_j \to \hbar n_j$), satisfying $n_+ + n_- = 1$. Similarly, Eq. (55) is equivalent to the dipole potential at weak coupling [32,33]:

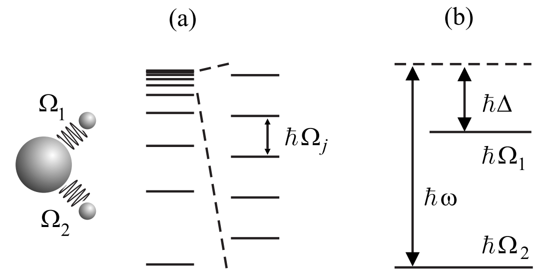

FIG. 2. (a) Schematic of a quasiclassical particle containing two harmonic oscillators with unperturbed frequencies Ω_1 and Ω_2. The energy spectrum for each individual oscillator at large energies (closeup on the right) is equidistant. The coupling with an oscillating field $\bar{E}$ near the beat resonance (54) is described by Eqs. (56). (b) A two-level system, with unperturbed eigenfrequencies Ω_1 and Ω_2, formally described by the same Eqs. (56), assuming the Rabi frequency $\Omega_{\mathrm{R}} = \varepsilon$.

$$\Phi = \frac{\hbar\Omega_{\mathrm{R}}^2}{4\Delta}(1 - 2n_1). \tag{65}$$

[At $n_1 = 0$, Eq. (65) is further analogous to Eq. (35), with the resonance at the transition frequency $\Omega_1 - \Omega_2$.]

This parallelism originates from Eq. (24) yielding Schrödinger-type equations for $\tilde{\mathcal{L}}_{\mathrm{int}}$ quadratic in internal mode amplitudes. Therefore for any classical particle governed by a hybrid potential, an analog is possible which is governed by a quantum dipole potential, and vice versa. Hence the two types of objects permit uniform manipulation techniques, as also discussed in Sec. VI.

VI. ONE-WAY WALLS

As an *effective* potential, Ψ can have properties distinguishing it from true potentials. Particularly, it can yield asymmetric barriers, or one-way walls [15–24], allowing current drive [34–36] and translational cooling [17,22,24,37,38]. We explain these barriers as follows.

Suppose a ponderomotive potential of the form (35) [or, similarly, (42)]. Given $\Omega = \Omega(z)$, with, say, $\partial_z \Omega < 0$, the average force $F_z = -\partial_z \Psi$ is everywhere in $+z$ direction, except at the exact resonance where the effective potential does not apply [Fig. 3(a)]. Hence particles can be transmitted when traveling in one direction but reflected otherwise, even assuming uniform $\bar{E}(z)$ [34–36]. In Ref. [15], such dynamics was confirmed for cyclotron-resonant rf fields, and, in Ref. [16], a similar scheme employing abrupt $\bar{E}(z)$ was proposed.

Hybrid potentials (Sec. V) permit yet another type of one-way walls. Assume uniform Ω_j and $\Delta > 0$; then Ψ [Eqs. (55) and (63)] is repulsive for cold particles ($J_1 < J_2$) but attractive for hot particles ($J_1 > J_2$). Thus if particles incident, say, from the left are preheated (via nonadiabatic interaction with another field), they will be transmitted, whereas those cold as incident from the right will be repelled [Fig. 3(b)]; hence the asymmetry.

In agreement with the parallelism shown in Sec. V C, similar one-way walls for cold atoms have been suggested [17–21,24,39] and enjoyed experimental verification [22,23].

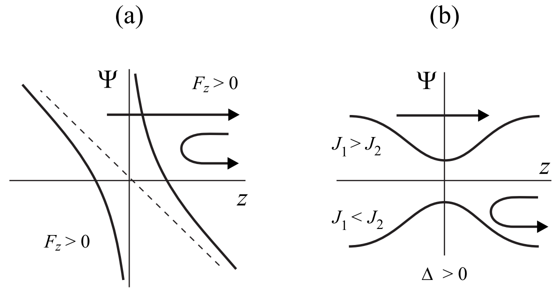

FIG. 3. Schematic showing two types of one-way walls (arrows denote particle transmission and reflection). (a) The one-way wall is due to a ponderomotive potential near a primary resonance [Eqs. (34) and (35)] with uniform $\partial_z\Omega < 0$, so $F_z = -\partial_z \Psi > 0$ for all z, except at the exact resonance where the effective potential does not apply. (The inclined asymptote corresponds to $\Psi = \Omega J + \text{const.}$) (b) The one-way wall is due to a hybrid potential [Eqs. (55) and (63)] with uniform Ω_j and $\Delta > 0$: particles incident from the left receive $J_1 > J_2$, so they see an attractive Ψ; those incident from the right have $J_1 < J_2$, so they see a repulsive Ψ.

However, the result reported here is that quasiclassical particles like Rydberg atoms and molecules can, in principle, be manipulated in the same manner, despite that their involved eigenspectrum is different (Fig. 2).

VII. CONCLUSIONS

We propose a method to calculate the generalized effective potential Ψ for a nonrelativistic classical particle undergoing arbitrary oscillations in high-frequency or static fields. We derive Ψ from the oscillation eigenfrequencies in the particle-field system [Eq. (6)], such as in the dressed-atom approach [11–14] but from nonlinear classical equations [Eqs. (24)]. Specifically, we show that the ponderomotive potential [Eqs. (35) and (A6); Fig. 1(a)] extended to a nonlinear oscillator has multiple branches near the primary resonance [Eq. (42); Fig. 1(d)]. We also show that, for a pair of natural frequencies in a beat resonance, Ψ scales linearly with the internal actions [Eqs. (55) and (63)] and is analogous to the dipole potential [Eqs. (64) and (65)] for a two-level quantum system (Fig. 2). Thus cold quantum particles and highly excited quasiclassical objects permit uniform manipulation tools, particularly, stationary asymmetric barriers, or one-way walls (Fig. 3).

ACKNOWLEDGMENTS

This work was supported by DOE Contract Nos. DE-FG02-06ER54851 and DE-FG02-05ER54838, and by the NNSA under the SSAA Program through DOE Research Grant No. DE-FG52-04NA00139.

APPENDIX: GENERAL EXPRESSION FOR THE PONDEROMOTIVE POTENTIAL

Consider generalization of the ponderomotive potential (34) and (35) to multiple internal oscillations and vector field

$$\bar{\mathbf{E}} = \sum_{\mu} \bar{\mathbf{E}}_{\mu}, \quad \bar{\mathbf{E}}_{\mu} = \mathbf{e}_{\mu} \bar{E}_{\mu}, \tag{A1}$$

composed of modes μ with polarizations $\mathbf{e}_{\mu}$. From Sec. IV A, it is known that the internal energy $\delta\boldsymbol{\Omega}\cdot\mathbf{J}$ yields a negligible contribution to Ψ. Thus

$$\Phi_0 = \sum_{\mu} \delta\omega_{\mu} I_{\mu}, \tag{A2}$$

where the infinitesimal frequency shifts $\delta\omega_{\mu}$ of the field are found as follows. Use [3]

$$\tilde{\mathcal{L}}_{\text{int}} = \tfrac{1}{2}\,\text{Re}[\bar{\mathbf{E}}^* \cdot \bar{\mathbf{d}}], \tag{A3}$$

where insignificant oscillating terms are removed, and $\bar{\mathbf{d}}$ is the particle induced dipole moment; then Eq. (24) rewrites as

$$-\delta\omega_{\mu}\rho_{\mu}\bar{E}_{\mu} = \tfrac{1}{4}\mathbf{e}_{\mu}^{*} \cdot \bar{\mathbf{d}}. \tag{A4}$$

Multiply Eq. (A4) by $\bar{E}_{\mu}^{*}$ and substitute $\bar{\mathbf{d}} = \hat{\alpha}\bar{\mathbf{E}}$, where $\hat{\alpha}$ is the polarizability tensor; then

$$\delta\omega_{\mu} I_{\mu} = -\tfrac{1}{4}\bar{\mathbf{E}}_{\mu}^{*} \cdot \hat{\alpha}\bar{\mathbf{E}}. \tag{A5}$$

Hence summation over all field modes yields the known expression [40,41]

$$\Phi_0 = -\tfrac{1}{4}(\bar{\mathbf{E}}^* \cdot \hat{\alpha}\bar{\mathbf{E}}), \tag{A6}$$

which holds for any internal modes contributing to $\hat{\alpha}$ [3].

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